

Modelling Spatial Time Series with Seasonal Regimes

Modellazione di Serie Spazio-Temporali con Regimi Stagionali

Valter Di Giacinto

Banca d'Italia – Filiale di L'Aquila, Nucleo regionale per la ricerca economica
Corso Federico II, 1, 67100 L'Aquila; e-mail valter.digiacinto@bancaditalia.it

Riassunto: Il presente lavoro estende la parametrizzazione della classe di modelli spazio-temporali STARMAG permettendo ai coefficienti di variare nei diversi periodi dell'anno. In tal modo il modello riesce a replicare le fluttuazioni stagionali dell'auto-correlazione che caratterizzano alcune tipologie di fenomeni, quale l'incidenza della criminalità di tipo mafioso, che costituisce il caso concreto analizzato nel lavoro.

Keywords: STARMAG model, seasonal regimes, spatial autocorrelation

1. Introduction ⁽¹⁾

While the statistical treatment of seasonal fluctuations of the level of a phenomenon has received large attention in the time series literature, it is only recently that the possibility that seasonality might induce more profound alteration in the dynamics of the observed processes has been investigated, leading to the introduction of statistical models allowing for seasonal specific regimes (Proietti, 2002; Kostov and Lingard, 2005). In a space-time environment we are interested in assessing if during the year, besides first moments, the second moments of the process, and namely spatial autocorrelation, display some cyclical pattern as well.

The aim of this paper is to develop a model capable of capturing such seasonal variation in the correlation structure of a given spatio-temporal process. To this purpose a new specification of the STARMA (Space Time AutoRegressive Moving Average) model class, as generalized in Di Giacinto (1994), is introduced by allowing model coefficients to take different values for each fraction of the year.

At the moment only seasonal variation in autocorrelation is dealt with, leaving the treatment of seasonal heteroskedasticity for future extensions. For the sake of conciseness seasonal variation in the level of the series, being a well known topic, is ruled out as well, and a zero mean process is assumed throughout.

In the next section model specification is detailed and maximum likelihood estimation is briefly discussed. Section 3 closes with an empirical application to the analysis of an indicator of crime incidence at the provincial level in Italy.

While the comparison of the performance of the proposed model with other approaches based on geostatistical structures or on the Kalman filter would be of clear interest, the limited space available force us to postpone the exercise to future communications.

¹ The views expressed are the author's and do not necessarily reflect those of the Banca d'Italia.

2. The model

Letting y_{it} denote the value of a spatial time series recorded on location i , ($i=1, \dots, N$) at time t , ($t=1, \dots, T$), and letting $s(t)$, ($s=1, \dots, S$), denote the season⁽²⁾ of the year corresponding to period t , the model, referred to as Seasonal Regimes STARMA Generalized (SR-STARMAG), has the following expression

$$y_{it} - \sum_{h=0}^p \sum_{k=m(h)}^{\lambda(h)} \phi_{hks} L_k y_{i,t-h} = u_{it} - \sum_{h=0}^q \sum_{k=m(h)}^{\nu(h)} \theta_{hks} L_k u_{i,t-h} \quad (1)$$

where $m(h)=1$ if $h=0$, $m(h)=0$ if $h>0$, u_{it} is a space-time homoskedastic white-noise process with autocovariance function $E(u_{it}u_{i+j,t+l}) = \sigma^2 \delta_{0j} \delta_{0l} - \delta_{ij}$ denoting the Kronecker's delta – and where L_k is the spatial lag operator of order k (see, e.g., Anselin and Smirnov, 1996, for a definition). Differently from the basic STARMAG specification in this case AR and MA coefficients are indexed by the season of the year. Stacking the spatial observations for a given time period the following vector expression for the model ensues

$$A_s(B)y_t = M_s(B)u_t \quad (2)$$

where

$$A_s(B) = (A_{0s} - A_{1s}B - \dots - A_{ps}B^p), \quad M_s(B) = (M_{0s} - M_{1s}B - \dots - M_{qs}B^q)$$

with B denoting the usual Box-Jenkins backward shift operator, and where $y_t = [y_{1t}, \dots, y_{Nt}]'$, $u_t = [u_{1t}, \dots, u_{Nt}]'$, $e(u_t u_{t-h}') = \delta_{0h} \sigma^2 I_N$. As usual I_N is the identity matrix of dimension N , while the $N \times N$ coefficients matrices A_h and M_h are so defined

$$A_{0s} = I_N - \sum_{k=1}^{\lambda_0} \phi_{0ks} W_k, \quad A_{hs} = - \sum_{k=0}^{\lambda_h} \phi_{hks} W_k; \quad M_{0s} = I_N - \sum_{k=1}^{\nu_0} \theta_{0ks} W_k, \quad M_{hs} = - \sum_{k=0}^{\nu_h} \theta_{hks} W_k.$$

with W_k denoting the k -th order spatial weights matrix. The conditional density of the random vector y_t is proportional to the density of the error process

$$f(y_t | y_{t-1}, \dots, y_{t-p}, u_{t-1}, \dots, u_{t-q}) = \|J\| f(u_t)$$

where $\|J\|$ denotes the jacobian of the transformation, and the log-density, assuming gaussian errors, is the following

$$\begin{aligned} \log f(y_t | y_{t-1}, \dots, y_{t-p}, u_{t-1}, \dots, u_{t-q}) &= \\ &= \text{const.} + \log \|A_{0s}\| - \log \|M_{0s}\| - \frac{N}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} u_t' u_t \end{aligned} \quad (3)$$

² The term season is used to denote generically any of the periods composing a year.

where $u_t = M_{0s}^{-1}(A_{0s}y_t - A_{1s}y_{t-1} - \dots - A_{ps}y_{t-p} + M_{1s}u_{t-1} + \dots + M_{qs}u_{t-q})$. The conditional log-likelihood for the whole series is simply the sum of single period log-likelihoods

$$ll(\phi, \theta, \sigma^2 | y_{t-1}, \dots, y_{t-p}, u_{t-1}, \dots, u_{t-q}) = \sum_{t=1}^T \log f(y_t | y_{t-1}, \dots, y_{t-p}, u_{t-1}, \dots, u_{t-q}). \quad (4)$$

First order conditions for the maximum, while yielding the usual ML estimator of the error variance

$$\hat{\sigma}^2 = \frac{1}{NT} \sum_{t=1}^T u_t' u_t \quad (5)$$

do not admit a closed form solution for the remaining parameters. Substituting (5) in (4) a concentrated likelihood obtains

$$ll_c(\phi, \theta | y_{t-1}, \dots, y_{t-p}, u_{t-1}, \dots, u_{t-q}) = const. + \sum_{s=1}^S T_s (\log \|A_{0s}\| - \log \|M_{0s}\|) - \frac{NT}{2} \log \left(\sum_{t=1}^T u_t' u_t \right)$$

where T_s denotes the number of times the s -th season of the year is included in the observed time series, that has to be maximized iteratively by means of an optimization procedure.

3. An empirical illustration

To provide an empirical illustration of the proposed modelling approach we analyse the monthly series of the incidence of a specific crime typology – offences classified as a specific form of organized crime (‘reati di stampo mafioso’) – on the Italian provinces over the period 1996-2003. The indicator is defined as the number of reported offences in a given month per 10.000 residents and, since both the level and variance of the process show great cross-sectional variation, the individual time series composing the spatial time series were standardized in a preliminary stage.

A prior exploratory analysis of the series showed how spatial autocorrelation is not constant over the year, peaking in the months of March and during the summer, reaching the maximum in August, an evidence apparently contradicting the time-stationarity assumption underlying the usual STARMAG modelling approach. As a preliminary step in model identification the inspection of the space-time autocorrelation function of the series, with spatial lags computed on a contiguity basis, suggested an autoregressive model including seasonal lags. After some specification search the following STARG model was selected

$$(A_0 - A_1 B - A_{12} B^{12}) y_t = u_t \quad (7)$$

with A matrices computed allowing for spatial lags up to order 1.

The first row of Table 1 reports the estimated coefficients of the STARG model, while the following rows give the estimates of the corresponding SR-STARG specification, separately for each month of the year, plus some goodness of fit statistics.

Table 1: *Estimation results*

Model	Estimated AR coefficients				
	Lag (0,1) (#)	Lag (1,0)	Lag (1,1)	Lag (12,0)	Lag (12,1)
STARG	0,133 **	0,179 **	0,045 *	0,079 **	0,146 **
SR-STARG					
January	0,214 **	0,180 **	-0,049	0,054	0,037
February	0,087 *	0,142 **	-0,043	0,080 *	0,132 **
March	0,169 **	0,213 **	0,086 *	0,095 **	0,176 **
April	0,088 *	0,145 **	-0,003	0,045	0,089 *
May	0,001	0,190 **	0,072 **	0,065	-0,045 *
June	0,198 **	0,214 **	0,075 *	0,087 **	0,241 **
July	0,252 **	0,157 **	0,011	0,063 **	0,293 **
August	0,255 **	0,158 **	0,216 **	0,072 **	0,077 **
September	0,018	0,183 **	0,048	0,007	-0,055 *
October	0,060	0,143 **	-0,041	0,028	0,107 **
November	-0,019	0,129 **	-0,006	0,056	0,144 **
December	0,018	0,181 **	0,066	0,079 *	0,170 **
Goodness of fit statistics					
	Observations	Log-likelihood	Pseudo-R ²	AIC	BIC
STARG	8.652	955,4	0,068	-1.896,8	-1.847,4
SR-STARG	8.652	1021,4	0,073	-1.918,8	-1.480,7

(#) The first number in brackets relates to the time lag and the second to spatial lag. * and ** denote significance at the 5% and 1% level, respectively.

As expected, estimated AR coefficients display a rather clear seasonal pattern. In particular the coefficients corresponding to spatially lagged variables attain the highest values in the same period where the monthly Moran indices showed seasonal peaks. A likelihood ratio test of the STARG model against the SR-STARG specification rejects the null of the absence of seasonal regimes with a p-value <0,0001 and the STARG-SR is identified as the best specification by the Akaike's information criterion, while the more penalizing BIC favours the restricted model. Overall the SR-STARMAG model class appears to perform well in this specific case, being able to replicate the seasonal pattern of spatial autocorrelation detected by the exploratory analysis.

References

- Anselin L., Smirnov O. (1996) Efficient Algorithms for Constructing Proper Higher Order Spatial Lag Operators, *Journal of Regional science*, 36, 67-89.
- Di Giacinto V. (1994) Su una generalizzazione dei modelli spazio-temporali autoregressivi media mobile (STARMAG), *Atti della XXXVII Riunione SIS*, San Remo, Vol. II, 35-42.
- Kostov P., Lingard J. (2005) Seasonal Specific Model Analysis of UK Cereal Prices, Economics Working Paper Archive EconWPA, No. 0507014.
- Proietti T. (2002) Seasonal Specific Structural Time Series Models, EUI Working Paper, No. 2002/10.